

# LEFT-INVARIANT SUB-RIEMANNIAN ENGEL STRUCTURES: ABNORMAL GEODESICS AND INTEGRABILITY

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**ABSTRACT.** We provide the first known family of examples of integrable sub-Riemannian structures admitting strictly abnormal geodesics. These examples were obtained through the analysis of the equivalence problem for sub-Riemannian Engel structures. We proved that 6 invariants define a sub-Riemannian Engel structure and described the classification of left-invariant sub-Riemannian structures of Engel type. As an application of these results we provide a criterion of strict abnormality of geodesics as well as estimates on conjugate times in terms of the obtained invariants.

## 1. INTRODUCTION

A four-dimensional manifold  $M$  with a two-dimensional distribution  $\mathcal{D}$  is called an Engel manifold, if  $\mathcal{D}$  satisfies the following non-integrability conditions

$$\begin{aligned}\operatorname{rank}([\mathcal{D}, \mathcal{D}]) &= 3, \\ \operatorname{rank}([\mathcal{D}, [\mathcal{D}, \mathcal{D}]]) &= 4\end{aligned}$$

where  $[\mathcal{D}, \mathcal{D}]$  consists of those tangent vectors which can be obtained by taking commutators of local sections of  $\mathcal{D}$ .

The fact that any two Engel structures are locally equivalent is attributed to Engel and Cartan [13, 15]. Their normal form theorem states that any Engel structure is locally diffeomorphic to  $\mathbb{R}^4$  with coordinates  $(x, y, p, q)$  and with  $\mathcal{D}$  annihilated by two one forms

$$\begin{aligned}\omega_1 &= dy - p dx, \\ \omega_2 &= dp - q dx.\end{aligned}$$

Engel structures have various applications in mechanics, geometry and topology [18, 22]. Our interest comes from the optimal control theory and, in particular, from sub-Riemannian geometry. A positive-definite metric  $g$  on  $\mathcal{D}$  turns  $M$  into a metric space with the distance

$$d(q_0, q_1) = \inf_{\gamma} \left\{ \int_0^1 \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt \right\}$$

where the infimum is taken over all Lipschitzian curves  $\gamma(t)$  such that  $\gamma(0) = q_0$ ,  $\gamma(1) = q_1$  and  $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$  for almost every  $t$ . Locally minimizing curves are called geodesics.

The local equivalence problem for sub-Riemannian Engel manifold is non-trivial: a simple dimensions argument implies that locally a sub-Riemannian metric depends on 3 functions. Section 2 of the paper is devoted to the study of local invariants of Engel structures. We construct the canonical frame and study the generating set of invariants using deformation arguments. Second cohomology spaces of Engel Lie algebra appear naturally there and allow us to show that an arbitrary Engel structure is defined by 6 invariants

$$T_i, \quad 1 \leq i \leq 6,$$

see Theorem 1 for details. Note that there are non-trivial differential relations between them which follow from the Jacobi identity.

In Subsection 2.4 we specialize these results to the left-invariant case. We obtain a description of the moduli space of left-invariant Engel structures and give a complete classification. This problem was previously considered in the works of Almeida [8, 9], but the classification there is incomplete.

We develop some important applications of obtained results in Section 3. We use the canonical frame to study abnormal geodesics of Engel structures, whose existence is a characteristic feature of sub-Riemannian geometry. A geodesic can be normal and abnormal at the same time, and if this not the case, we call such a curve *strict*. Theorem 4 gives an easy test for checking the strictness of abnormal geodesics: an abnormal geodesic is strict if and only if  $T_4 \neq 0$  along the geodesic.

Another application is the comparison theorem for abnormal geodesics on an Engel manifold. Theorem 5 relates local minimality of an abnormal geodesic with the behavior of

$$(1) \quad \Delta = T_6 + \frac{1}{2}\dot{T}_2 - \frac{1}{4}(T_2)^2$$

along the geodesic. In the left invariant case we can formulate a sharp result that all conjugate times are:

$$(2) \quad t_{conj} = \frac{\pi k}{\sqrt{\Delta}}, \quad \forall k \in \mathbb{Z}_+.$$

Abnormal geodesics have been studied extensively [6, 12, 17, 21, 23], but many open questions remain. One problem of the particular interest is a qualitative behavior of a sub-Riemannian sphere in the neighbourhood of an abnormal extremal [1]. Some studies regarding this problem are done for the Martinet structure [3] and the flat Engel structure [10, 11]. These models do not admit strictly abnormal geodesics. Examples admitting strictly abnormal geodesics were considered in [12], unfortunately the models presented there are not integrable. This motivates the last section of our paper. We examine which left-invariant structures admit integrable geodesic flows. We study some particularly nice examples admitting strictly abnormal geodesics and integrate the vertical part of the Hamiltonian system for normal geodesics. This allows in principle to obtain an explicit parametrization of sub-Riemannian spheres. However this goes beyond the scope of the paper.

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## 2. SUB-RIEMANNIAN ENGEL STRUCTURES AS LOCAL DEFORMATIONS OF ENGEL GROUP

It is well known that a sub-Riemannian Engel structure can be endowed with a canonical global frame. We remind the construction for the convenience.

**2.1. Canonical frame for Engel sub-Riemannian structure.** Let  $\mathcal{E}$  and  $\mathcal{V}$  be distributions on an arbitrary manifold. We denote by  $[\mathcal{E}, \mathcal{V}]$  a distribution which is generated by brackets of germs of sections of  $\mathcal{E}$  and  $\mathcal{V}$ . For arbitrary distribution  $\mathcal{D}$  we use a notation  $\mathcal{D}^1 = \mathcal{D}$  and  $\mathcal{D}^i = [\mathcal{D}, \mathcal{D}^{i-1}]$ .

**Definition 1.** A *Levi form*  $\mathcal{L}$  on arbitrary distribution  $\mathcal{E}$  is a bilinear function which depends on a point:

$$\mathcal{L}_p : \wedge^2 \mathcal{E}_p \rightarrow T_p M / \mathcal{E}_p.$$

For arbitrary vectors  $v, w \in T_p M$  it is defined by:

$$\mathcal{L}_p(v, w) = [V, W]_p \mod \mathcal{E}$$

where  $V$  and  $W$  are such sections of  $\mathcal{E}$  that  $V_p = v$  and  $W_p = w$ .

It is straightforward to check that the definition of the Levi form doesn't depend on the chosen sections  $V$  and  $W$ .

**Lemma 1.** *Let  $\mathcal{D}$  be an Engel distribution and  $\mathcal{L}$  be the Levi form on  $\mathcal{D}^2$ . Then the kernel  $K$  of  $\mathcal{L}$  is one-dimensional and is contained in  $\mathcal{D}$ .*

*Proof.* First of all  $\mathcal{L} \neq 0$  since  $TM = [\mathcal{D}, \mathcal{D}^2] \subseteq [\mathcal{D}^2, \mathcal{D}^2]$ . Therefore the kernel is 1-dimensional. Assume that  $K \not\subseteq \mathcal{D}$ . Then  $\mathcal{D}^2 = K \oplus \mathcal{D}$  and  $[\mathcal{D}, \mathcal{D}^2] = [\mathcal{D}, \mathcal{D} + K] = [\mathcal{D}, \mathcal{D}] = \mathcal{D}^2$  which contradicts the definition of an Engel structure.  $\square$

*Remark 1.* The kernel of  $\mathcal{L}$  in fact defines a characteristic line field of the distribution  $\mathcal{D}$ . Its integral lines are abnormal geodesics of arbitrary sub-Riemannian structure  $(g, \mathcal{D}, M)$ .

Let  $K$  be a kernel of Levi form  $\mathcal{L}$ . With every 4-dimensional Engel structure we can associate a canonical up to an action of a discrete group frame. Namely, let  $X_2$  be one of the two orthonormal vectors in  $K$ . Let  $X_1$  be an orthogonal complement to  $X_2$ . Then vectors  $X_3$  and  $X_4$  are defined as follows:

$$X_3 = [X_1, X_2], X_4 = [X_1, X_3].$$

The frame  $\{X_1, X_2, X_3, X_4\}$  is unique up to the action of a group  $T = \mathbb{Z}_2 \times \mathbb{Z}_2$  which is generated by

$$(3) \quad \begin{aligned} T_1: \{X_1, X_2, X_3, X_4\} &\rightarrow \{-X_1, X_2, -X_3, X_4\}, \\ T_2: \{X_1, X_2, X_3, X_4\} &\rightarrow \{X_1, -X_2, -X_3, -X_4\}. \end{aligned}$$

Note that we can omit the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  ambiguity by fixing the frame orientation as well as the orientation of  $\mathcal{D}$ .

**2.2. Structure function.** Every sub-Riemannian Engel structure induces a canonical filtration of the tangent bundle of  $M$ :

$$(4) \quad 0 = F^0 \subset F^{-1} \subset F^{-2} \subset F^{-3} \subset F^{-4} = TM$$

where  $F^{-1} = \mathbb{R}X_1$ ,  $F^{-2} = \mathcal{D}$  and  $F^{-3} = \mathcal{D}^2$ .

An arbitrary filtration  $F$  of the tangent bundle defines a structure of Lie algebra on  $T_p M$  for every point  $p$ . The Lie algebra structure is induced from the Lie bracket of vector fields. If  $v \in F_p^{-i}$ ,  $w \in F_p^{-j}$  and  $V \in \Gamma(F^{-i})$ ,  $W \in \Gamma(F^{-j})$  are such vector fields that  $V_p = v$ ,  $W_p = w$  then

$$(5) \quad [v, w] = [V, W]_p \mod F^{-i-j+1}$$

The definition above doesn't depend on the choice of vector fields  $V$ ,  $W$ . Indeed, if  $\{X_\alpha\}$ ,  $\{Y_\beta\}$  are local frames for  $F^{-i}$  and  $F^{-j}$  and  $V = f^\alpha X_\alpha$ ,  $W = g^\beta Y_\beta$  then

$$[V, W] = f^\alpha g^\beta [X_\alpha, Y_\beta] + V(g^\beta)Y_\beta - W(f^\alpha)X_\alpha \equiv f^\alpha g^\beta [X_\alpha, Y_\beta] \mod F^{-i-j+1}$$

**Proposition 1.** *The associated graded Lie algebra  $\text{gr } F = \bigoplus_i F^{-i}/F^{-i+1}$  is isomorphic to the Engel Lie algebra given by*

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4.$$

*Proof.* We check that an identification

$$\begin{aligned} X_1 &\rightarrow e_1, \\ X_2 + \text{span}\{X_1\} &\rightarrow e_2, \\ X_3 + \text{span}\{X_1, X_2\} &\rightarrow e_3, \\ X_4 + \text{span}\{X_1, X_2, X_3\} &\rightarrow e_4 \end{aligned}$$

gives the required isomorphism. Indeed

$$\begin{aligned} [X_1, X_2 + \alpha X_1] &= X_3 + X_1(\alpha)X_1, \\ [X_1, X_3 + \beta X_2 + \alpha X_1] &= X_4 + \beta X_3 + X_1(\beta)X_2 + X_1(\alpha)X_1, \end{aligned}$$

□

Engel algebra is a graded Lie algebra with grading  $\deg(e_i) = -i$ . From Proposition 1 we see that Lie brackets of the canonical frame are deformations of Engel Lie algebra brackets. In other words

$$(6) \quad [X_i, X_j] = X_{i+j} + C_{ij}^k X_k, \quad i < j$$

where  $X_i = 0$  if  $i > 4$ . If we consider the frame  $\{X_i\}$  as an isomorphism from Engel Lie algebra  $\mathfrak{g}$  with the basis  $e_i$  to  $T_p M$  then  $C_{ij}^k$  defines a linear map from  $\wedge^2 \mathfrak{g}$  to  $\mathfrak{g}$

$$C(e_i, e_j) = C_{ij}^k e_k$$

for every point  $p \in M$ .

**Definition 2.** We call  $C: M \rightarrow \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$  a *structure function* of a sub-Riemannian Engel structure.

Proposition 1 guaranties that  $C$  has a positive degree i.e.  $C_{ij}^k = 0$  if  $i + j - k \leq 0$ .

**2.3. Equivalence problem.** Now we relate an equivalence problem for sub-Riemannian Engel structures with certain cohomologies of Engel Lie algebra. To fix notations for cohomology groups consider the space

$$C^k(\mathfrak{g}, V) = \text{Hom}(\wedge^k \mathfrak{g}, V)$$

of  $k$ -cochains on  $\mathfrak{g}$  with values in  $\mathfrak{g}$ -module  $V$ . Then the coboundary operator

$$\partial: C^k(\mathfrak{g}, V) \rightarrow C^{k+1}(\mathfrak{g}, V)$$

is defined by

$$\begin{aligned} \partial \alpha(v_0, v_1, \dots, v_k) &= \sum_{i=0}^k (-1)^i v_i \cdot \alpha(v_0, \dots, \hat{v}_i, \dots, v_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([v_i, v_j], v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k). \end{aligned}$$

We denote the cohomology groups of the complex  $C^k(\mathfrak{g}, V)$  by  $H^k(\mathfrak{g}, V)$ . In order to compute  $\partial$  explicitly it is sufficient to evaluate  $\partial$  on  $\mathfrak{g}^*$  and  $V$ , and use the fact that  $\partial$  is a differentiation. In our particular case  $H^\bullet(\mathfrak{g}, \mathfrak{g})$  we have:

$$\begin{aligned} \partial(e_3^*) &= -e_1^* \wedge e_2^*, \\ \partial(e_4^*) &= -e_1^* \wedge e_3^*, \\ \partial(e_1) &= -e_3 \otimes e_2^* - e_4 \otimes e_3^*, \\ \partial(e_2) &= e_3 \otimes e_1^*, \\ \partial(e_3) &= e_4 \otimes e_1^*. \end{aligned}$$

Since the structure function has always positive degree it is sufficient to consider the sub-complex  $C^k(\mathfrak{g}, V)_+$  of positively graded chains together with the sub-groups  $H^k(\mathfrak{g}, V)_+$  of positively graded cohomologies.

**Proposition 2.** *Vector fields  $X_i$ ,  $1 \leq i \leq 4$  define a normal frame for sub-Riemannian Engel structure iff the structure function  $C$  belongs to the subspace  $N \subset \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})_+$  which is defined by*

$$(7) \quad 0 = C_{12}^2 = C_{12}^1,$$

$$(8) \quad 0 = C_{13}^3 = C_{13}^2 = C_{13}^1,$$

$$(9) \quad 0 = C_{23}^4.$$

Moreover  $N$  satisfies

$$(10) \quad N \oplus \text{im } \partial_+ = \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})_+$$

where  $\text{im } \partial_+$  is a positively graded part of  $\text{im } \partial$ .

*Proof.* Equalities (7) follows from  $[X_1, X_2] = X_3$ , (8) follows from  $[X_1, X_3] = X_4$  and (9) follows from the fact that  $X_2$  generates the kernel of the Levi form on  $\mathcal{D}^2$ . To prove the second statement of the Proposition note that  $\text{im } \partial_+$  is 6-dimensional. In notations  $e_i^{jk} = e_i \otimes e_j^* \wedge e_k^*$  and  $e_i^j = e_i \otimes e_j^*$  the vector space  $\text{im } \partial_+$  is generated by:

$$\begin{aligned} \partial(e_1^2) &= e_4^{23} \\ \partial(e_1^3) &= -e_1^{12} - e_3^{23} \\ \partial(e_1^4) &= -e_1^{13} - e_3^{24} - e_4^{34} \\ \partial(e_2^3) &= -e_2^{12} + e_3^{13} \\ \partial(e_2^4) &= -e_2^{13} + e_3^{14} \\ \partial(e_3^4) &= -e_1^{13} + e_4^{14} \end{aligned}$$

We see that indeed (10) holds.  $\square$

Next important observation is that the structure function  $C$  could be reconstructed from  $\ker \partial$  part of  $C$ . Let  $H = \ker \partial \cap N$  and  $B$  be an arbitrary complementary vector space

$$B \oplus H = N.$$

The splitting above defines decomposition  $C = C_H + C_B$ . Note that since  $N \oplus \text{im } \partial_+ = \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})_+$  the space  $H$  represents elements from the second cohomologies  $H^2(\mathfrak{g}, \mathfrak{g})$ .

**Proposition 3.**  *$C_B$  part of the structure function  $C$  can be recovered from the  $C_H$  part using differentiation along canonical vector fields  $X_i$ .*

*Proof.* We split the Lie bracket of vector fields into two parts:

$$[X_i, X_j] = [X_i, X_j]_{\mathfrak{g}} + C_{ij}^k X_k,$$

where  $[\cdot, \cdot]_{\mathfrak{g}}$  is the Engel Lie algebra bracket. Then for two arbitrary vector fields  $Y = \alpha^i X_i$  and  $Z = \beta^j X_j$  we have:

$$(11) \quad [Y, Z] = \alpha^i \beta^j [X_i, X_j]_{\mathfrak{g}} + \alpha^i \beta^j C_{ij}^k X_k + \alpha^i X_i(\beta^j) X_j - \beta^j X_j(\alpha^i) X_i.$$

In this proof we denote by  $\{ \}$  the cyclic sum in  $(i, j, k)$ . Then using (11) the Jacobi identity reads as:

$$\begin{aligned} 0 &= \{ [[X_i, X_j], X_k] \} = \{ [[X_i, X_j]_{\mathfrak{g}}, X_k] \} + \{ [C_{ij}^l X_l, X_k] \} \\ &= \{ [[X_i, X_j]_{\mathfrak{g}}, X_k]_{\mathfrak{g}} + C([X_i, X_j]_{\mathfrak{g}}, X_k) \} + \{ [C_{ij}^l X_l, X_k] \} \\ &= \{ C([X_i, X_j]_{\mathfrak{g}}, X_k) \} + \{ C_{ij}^l [X_l, X_k] - X_k(C_{ij}^l) X_l \} \\ &= \{ C([X_i, X_j]_{\mathfrak{g}}, X_k) + C_{ij}^l [X_l, X_k]_{\mathfrak{g}} + C_{ij}^l C_{lk}^t X_t - X_k(C_{ij}^l) X_l \} \\ &= -\partial(C)(X_i, X_j, X_k) + C \circ C(X_i, X_j, X_k) + D_X(C)(X_i, X_j, X_k). \end{aligned}$$

Here

$$\partial(C)(X_i, X_j, X_k) = C([X_i, X_j]_{\mathfrak{g}}, X_k) + C_{ij}^l [X_l, X_k]_{\mathfrak{g}}$$

is the Lie algebra differential of  $C \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ ,

$$C \circ C(X_i, X_j, X_k) = \{ C(C(X_i, X_j), X_k) \}$$

is a cyclic composition of  $C$  with itself and

$$D_X(C)(X_i, X_j, X_k) = \{ -X_k(C(X_i, X_j)) \}$$

is a cyclic covariant derivative of  $C$ . To sum up we obtain an equality

$$(12) \quad \partial(C) = C \circ C + D_X(C).$$

Let us fix an arbitrary positive integer number  $d$ . We compute the  $d$ -degree component of the equality (12). First

$$(C \circ C)_d = \sum_{i+j+k-t=d} C_{ij}^l C_{lk}^t e_t^{ijk}.$$

Since the degree of the coefficients of  $C$  is greater than 0 and  $(i+j-l)+(l+k-t) = d$  we have  $i+j-l < d$ ,  $l+k-t < d$  and  $(C \circ C)_d$  depends only on lower degree coefficients of  $C$ . Second, coefficients of degree  $d$  for  $D_X(C)$  are

$$(13) \quad D_X(C)_d = \sum_{i+j+k-l=d} -X_k(C_{ij}^l) e_l^{ijk}$$

and since  $i+j-l < d$  in (13) we obtain that  $D_X(C)_d$  depends only on lower degree coefficients.

Finally, the Lie algebra differential  $\partial: \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g})$  preserves the degree and is a monomorphism on the subspace  $B$ . On the other hand  $D_X(C)_d$  and  $(C \circ C)_d$  depends only on lower degree coefficients of  $C$ . Therefore  $\partial(C_B)$  can be expressed through the lower degree coefficients of  $C$  and differentiations along  $X_i$ . This means that in fact all coefficients in  $C_B$  can be expressed through  $C_H$ .  $\square$

We obtained that coefficients of  $C$  corresponding to  $H^2(\mathfrak{g}, \mathfrak{g})$  defines the whole structure function. The next theorem sums up the discussion of the section and describes explicitly the structure function of the normal frame.

**Theorem 1.** *For every oriented sub-Riemannian Engel structure  $g, \mathcal{D}, M$  there exists a canonical frame  $\{X_1, X_2, X_3, X_4\}$  given by conditions*

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] \in \text{span}\{X_1, X_2, X_3\}.$$

*The structure equations of the canonical frame are of the following form:*

$$(14) \quad \begin{aligned} [X_1, X_2] &= X_3, [X_1, X_3] = X_4, \\ [X_1, X_4] &= \frac{1}{2} A_{14}^1 X_1 + T_5 X_2 + T_3 X_3 + T_1 X_4 \\ [X_2, X_3] &= T_6 X_1 + T_4 X_2 + T_2 X_3 \\ [X_2, X_4] &= X_1(T_6) X_1 + X_1(T_4) X_2 + (T_4 + X_1(T_2)) X_3 + T_2 X_4 \\ [X_3, X_4] &= A_{34}^1 X_1 + A_{34}^2 X_2 - \frac{1}{2} A_{34}^3 X_3 + (T_4 + A_{34}^4) X_4, \end{aligned}$$

where

$$\begin{aligned} A_{34}^4 &= 2X_1(T_2) - X_2(T_1), \\ A_{14}^1 &= T_1T_4 + T_1X_1(T_2) - 3X_1(T_4) + X_2(T_3) + X_3(T_1) - X_1^2(T_2), \\ A_{34}^3 &= T_1T_4 + T_1X_1(T_2) - X_1(T_4) + X_2(T_3) - X_3(T_1) - X_1^2(T_2), \\ A_{34}^2 &= T_2T_5 - T_3T_4 - T_1X_1(T_4) - X_2(T_5) + X_1^2(T_4), \\ A_{34}^1 &= \frac{1}{2}T_2A_{14}^1 - T_6T_3 - T_1X_1(T_6) - \frac{1}{2}X_2(A_{14}^1) + X_1^2(T_6). \end{aligned}$$

In particular, the structure constants depends only on  $T_i$ ,  $1 \leq i \leq 6$  and their derivatives along  $X_i$ .

*Proof.* The basis of  $\ker \partial \cap N$  is represented by

$$\begin{aligned} H_1 &= e_4^{14}, \quad H_2 = e_3^{23} + e_4^{24}, \quad H_3 = e_3^{14}, \\ H_4 &= e_2^{23} + e_3^{24} + e_4^{34}, \quad H_5 = e_2^{14}, \quad H_6 = e_1^{23} \end{aligned}$$

with the following degrees of elements:

$$\begin{aligned} \deg(H_1) &= 1, \quad \deg(H_2) = \deg(H_3) = 2, \\ \deg(H_4) &= \deg(H_5) = 3, \quad \deg(H_6) = 4. \end{aligned}$$

We write  $C_H$  part of the structure function as

$$C_H = \sum_{i=1}^6 T_i H_i.$$

According to the proof of Proposition 3 we can express all other coefficients of the structure function using Jacobi identity (12) and proceeding degree by degree. We denote by  $C_{\leq i}$  the part of the structure function which includes elements of degree not greater than  $i$ .

A trivial consequence of Proposition 3 is that  $C_{\leq 1} = (C_H)_{\leq 1} = T_1 H_1$ . To find degree 2 components of the structure function we in accordance with (12) compute that  $H_1 \circ H_1 = 0$  and  $D_X(T_1 H_1)_{\leq 2} = 0$ . This implies that apart from  $C_H$  there are no other terms in degree 2, i.e.

$$C_{\leq 2} = (C_H)_{\leq 2} = T_1 H_1 + T_2 H_2 + T_3 H_3.$$

It is straightforward to check that  $C_{\leq 2} \circ C_{\leq 2} = 0$ . Applying (12) we obtain

$$\begin{aligned} 0 &= \partial(C_{\leq 3}) - (D_X(C_{\leq 2}))_{\leq 3} = \\ &= (C_{24}^3 - X_1(T_2))e_3^{123} + (X_2(T_1) - X_1(T_2) + C_{34}^4 - C_{24}^3)e_4^{124} \end{aligned}$$

which gives

$$\begin{aligned} C_{24}^3 &= X_1(T_2) \\ C_{34}^4 &= 2X_1(T_2) - X_2(T_1). \end{aligned}$$

Continuing computations we obtain the expression of all structure function coefficients through  $T_i$ . □

**2.4. Classification of 4-dimensional left-invariant Engel structures.** Now we consider the classification problem for 4-dimensional left-invariant Engel sub-Riemannian structures on Lie groups. The structure function is constant in this case. The following general form of the structure equations for the canonical left-invariant frame is a direct consequence of Theorem 1.

**Proposition 4.** *Let  $\{X_1, X_2, X_3, X_4\}$  be a canonical left-invariant frame for an Engel structure on a Lie group. Then the structure equations of the frame are:*

$$\begin{aligned}
 (15) \quad & [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \\
 & [X_1, X_4] = \frac{1}{2}AX_1 + T_5X_2 + T_3X_3 + T_1X_4, \\
 & [X_2, X_3] = T_6X_1 + T_4X_2 + T_2X_3, \\
 & [X_2, X_4] = T_4X_3 + T_2X_4, \\
 & [X_3, X_4] = CX_1 + BX_2 - \frac{1}{2}AX_3 + T_4X_4,
 \end{aligned}$$

where  $A = T_1T_4$ ,  $B = T_2T_5 - T_3T_4$ ,  $C = \frac{1}{2}AT_2 - T_3T_6 = \frac{1}{2}T_1T_2T_4 - T_3T_6$ .

Substituting the structure constants from Proposition 4 into the Jacobi formula we obtain a system of restrictions on  $T_i$ :

$$\begin{aligned}
 (16) \quad & 0 = T_1T_6 + 2T_2T_4, \\
 & 0 = T_1^2T_4 + 4T_2T_5, \\
 & 0 = T_1T_3T_4 - T_1T_2T_5 + 2T_4T_5, \\
 & 0 = T_1T_4^2 - T_1^2T_2T_4 + 2T_1T_3T_6 + 2T_5T_6, \\
 & 0 = T_1T_4^2 + 4T_2^2T_5 - 4T_2T_3T_4 + 2T_5T_6, \\
 & 0 = T_1T_2^2T_4 + T_1T_4T_6 - 2T_2T_3T_6.
 \end{aligned}$$

Solving the system above we get the classification of left-invariant sub-Riemannian Engel structures.

**Theorem 2.** *Any left-invariant sub-Riemannian Engel structure is locally equivalent to the structure from one of the following 5 classes. Any structure is locally uniquely defined by its invariants  $T_i$ .*

(1)  $\{T_2 = T_4 = T_6 = 0\}$ . Structure equations are

$$\begin{aligned}
 & [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \\
 & [X_1, X_4] = T_5X_2 + T_3X_3 + T_1X_4.
 \end{aligned}$$

(2)  $\{T_4 = T_6 = T_5 = 0\}$ . Structure equations are

$$\begin{aligned}
 & [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \\
 & [X_1, X_4] = T_3X_3 + T_1X_4, \\
 & [X_2, X_3] = T_2X_3, \quad [X_2, X_4] = T_2X_4
 \end{aligned}$$

(3)  $\{T_1 = T_2 = T_5 = 0\}$ . Structure equations are

$$\begin{aligned}
 & [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \\
 & [X_1, X_4] = T_3X_3, \quad [X_2, X_4] = T_4X_3, \\
 & [X_2, X_3] = T_6X_1 + T_4X_2, \\
 & [X_3, X_4] = -T_6T_3X_1 - T_4T_3X_2 + T_4X_4.
 \end{aligned}$$

(4)  $\{T_1 = T_3 = T_4 = T_5 = 0\}$ . Structure equations are

$$\begin{aligned}
 & [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \\
 & [X_2, X_4] = T_2X_4, \\
 & [X_2, X_3] = T_6X_1 + T_2X_3.
 \end{aligned}$$



- (5)  $\{T_4 = \frac{1}{2} \frac{T_2(T_1^2+4T_3)}{T_1}, T_5 = -\frac{1}{8}T_1^3 - \frac{1}{2}T_1T_3, T_6 = -\frac{T_2^2(T_1^2+4T_3)}{T_1^2}; T_1 \neq 0\}$ . The resulting structure equations are too long to list them here. Interested reader could write them down using Proposition 4.

*Remark 2.* As one can see families from the theorem above have non-trivial intersections.

*Remark 3.* An attempt to classify left-invariant Engel structures was made in [9]. Unfortunately the classification there is incomplete, since families 4 and 5 are missing. The mistake is in the second line of the proof of Theorem 4. It is claimed that  $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \cap [\mathfrak{p}, \mathfrak{p}] = 0$  where  $\mathfrak{p}$  is the subspace corresponding to the left-invariant Engel type distribution (i.e.  $\mathfrak{p}$  is generated by  $X_1, X_2$  in the notations of this article). This is not true as one can see using case 4 with  $T_6 = 0$ , where  $[X_2, [X_1, X_2]] = T_2X_3 \in [\mathfrak{p}, \mathfrak{p}]$ .

### 3. MINIMALITY OF GEODESICS ON ENGEL MANIFOLDS

The problem of finding minimal curves between  $q_0, q_T \in M$  is equivalent to the optimal control problem

$$(17) \quad \dot{q} = u_1X_1(q) + u_2X_2(q), \quad (u_1, u_2) \in \mathbb{R}^2, q \in M,$$

$$(18) \quad q(0) = q_0, \quad q(T) = q_T,$$

$$(19) \quad l(q) = \int_0^T \sqrt{u_1^2 + u_2^2} dt \rightarrow \min,$$

where  $u_i$  are the control parameters. It is well known that the minimum exists [2] and instead of minimizing  $l(q)$  one can minimize the action functional

$$(20) \quad J(q) = \int_0^T \frac{u_1^2 + u_2^2}{2} dt \rightarrow \min$$

with  $T$  fixed [2].

We use the Pontryagin's maximum principle (PMP) in order to derive geodesic equations. Consider the cotangent bundle  $T^*M$  and coordinate functions

$$h_i = \langle \lambda, X_i \rangle, \quad \lambda \in T_q^*M,$$

on the fibers of  $T^*M$ . The Hamiltonian of the maximum principle is a family of affine on fibres functions

$$H_u(\lambda, \nu) = \langle \lambda, u_1X_1 + u_2X_2 \rangle + \frac{\nu}{2}(u_1^2 + u_2^2) = u_1h_2 + u_2h_1 + \frac{\nu}{2}(u_1^2 + u_2^2).$$

**Theorem 3** (PMP, [5]). *If a pair  $(\tilde{u}(t), \tilde{q}(t))$  is optimal, then there exists a Lipschitzian curve  $\lambda_t \in T_{\tilde{q}(t)}^*M$  and a number  $\nu \leq 0$ , s.t. the following conditions are satisfied*

- (1)  $\langle \lambda_t, \nu \rangle \neq 0$ ;
- (2)  $\dot{\lambda}_t = \vec{H}_{\tilde{u}(t)}$ ;
- (3)  $H_{\tilde{u}(t)} = \max_{u \in \mathbb{R}^2} H_u(\lambda_t, \nu)$ .

In this setting the Hamiltonian system of PMP is given by

$$\begin{aligned} \dot{q} &= u_1X_1(q) + u_2X_2(q), \\ \dot{h}_i &= \{H_u, h_i\}. \end{aligned}$$

where the Lie-Poisson bracket of vertical coordinate functions  $h_i$  depends only on the structure constants of  $X_i$ :

$$\{h_i, h_j\} = \langle \lambda, [X_i, X_j] \rangle = C_{ij}^k(q)h_k.$$

Using the structure equations (14) and the Leibniz rule we obtain

$$\begin{aligned}
 \dot{q} &= u_1 X_1(q) + u_2 X_2(q), \\
 \dot{h}_1 &= -u_2 h_3, \\
 \dot{h}_2 &= u_1 h_3, \\
 \dot{h}_3 &= u_1 h_4 + u_2 (T_6 h_1 + T_4 h_2 + T_2 h_3), \\
 \dot{h}_4 &= u_1 \left( \frac{1}{2} A_{14}^1 h_1 + T_5 h_2 + T_3 h_3 + T_1 h_4 \right) \\
 &\quad + u_2 (X_1(T_6) h_1 + X_1(T_4) h_2 + (T_4 + X_1(T_2)) h_3 + T_2 h_4).
 \end{aligned}
 \tag{21}$$

**Definition 3.** If an extremal satisfies the PMP with  $\nu = 0$  it is called *abnormal*, otherwise we say that it is *normal*. A geodesic is said to be strictly abnormal (strictly normal) if it is not a projection of some normal (abnormal) extremal.

To find the controls  $(u_1, u_2)$  we use condition (3) from Theorem 3. When  $\nu = 0$  the Hamiltonian is of the form

$$H_u = u_1 h_1 + u_2 h_2.$$

The only possibility for the maximum to be attained is  $h_1 \equiv h_2 \equiv 0$ . This implies  $u_1 h_3 = u_2 h_3 = 0$ . Since we are interested in curves with non-zero controls we obtain that  $h_3 \equiv 0$ . Then either  $u_1 \equiv 0$  or  $h_4 \equiv 0$ , but the non-triviality condition 1 of PMP in our case is equivalent to  $h_4 \neq 0$ . Therefore  $u_1 \equiv 0$  and abnormal geodesics must be integral curves of  $X_2$ . Along those curves the last equation reduces to

$$\dot{h}_4 = u_2 T_2 h_4$$

which solutions are sign definite for non-zero initial data. Therefore the non-triviality condition is satisfied and they are indeed abnormal geodesics.

Let us consider the case  $\nu \neq 0$ . Without loss of generality we can normalize  $(\lambda, \nu)$  in such a way that  $\nu = -1$ . Then the maximum is achieved when

$$\frac{\partial H_u}{\partial u_i} = h_i - u_i = 0 \iff u_i = h_i$$

Using the obtained controls in (21) we get

$$\begin{aligned}
 \dot{q} &= h_1 X_1(q) + h_2 X_2(q), \\
 \dot{h}_1 &= -h_2 h_3, \\
 \dot{h}_2 &= h_1 h_3, \\
 \dot{h}_3 &= h_1 h_4 + h_2 (T_6 h_1 + T_4 h_2 + T_2 h_3), \\
 \dot{h}_4 &= h_1 \left( \frac{1}{2} A_{14}^1 h_1 + T_5 h_2 + T_3 h_3 + T_1 h_4 \right) \\
 &\quad + h_2 (X_1(T_6) h_1 + X_1(T_4) h_2 + (T_4 + X_1(T_2)) h_3 + T_2 h_4).
 \end{aligned}
 \tag{22}$$

which is a Hamiltonian system with a Hamiltonian

$$H_{\tilde{u}(t)} = H = \frac{h_1^2 + h_2^2}{2}.$$

Let us find the structures admitting strictly abnormal geodesics.

**Theorem 4.** *An abnormal geodesic of an Engel sub-Riemannian structure is strictly abnormal if and only if  $T_4 \neq 0$  along the geodesic.*

*Proof.* Assume that an abnormal geodesic  $(q(t), h(t))$  satisfies (22). Since it is an integral curve of  $X_2$  we must have  $h_1 \equiv 0$ . Moreover, Hamiltonian  $H$  is a first integral of the system. Therefore  $H = h_1^2 + h_2^2 = \text{const} \neq 0$  and so  $h_2 = \text{const} \neq 0$ .

Thus from (22) it follows that  $h_3 \equiv 0$ . But the third equation gives us  $T_4 h_2^2 = 0$ , which can hold if and only if  $T_4 = 0$ . All these conditions reduce the system to the equation

$$(23) \quad \dot{h}_4 = h_2 T_2 h_4$$

which always has a solution.

On the other hand if  $T_4 = 0$  for an abnormal extremal we can reparametrize it in such a way that  $\dot{q} = X_2$ . By substituting  $h_1 = 0$ ,  $h_2 = 1$ ,  $h_3 = 0$  into (22) we reduce the system to (23). This equation always has a solution which guaranties that the abnormal extremal is normal as well.  $\square$

Any abnormal geodesic on sub-Riemannian Engel manifold is locally a minimizer. This follows from the fact that any strictly abnormal extremal for which

$$(24) \quad \lambda(t) \in (\mathcal{D}^2)^\perp \setminus (\mathcal{D}^3)^\perp$$

is locally minimizing [2]. For Engel manifold (24) is obviously true since  $(\mathcal{D}^3)^\perp = 0$  and  $\lambda(t) \in (\mathcal{D}^2)^\perp$ .

**Definition 4.** A geodesic  $\gamma_t$  connecting  $\gamma_0 = q_0$  with  $\gamma_T = q_T$  is called a  $C^0$ -local minimizer if there are no other geodesics connecting  $q_0$  with  $q_T$  in a sufficient small  $C^0$ -neighbourhood of  $\gamma_t$ .

To determine whether a given extremal curve is a  $C^0$ -local minimizer we use the notion of conjugate points. The definition is a bit different for normal and abnormal geodesics, but one can usually characterize them as solutions of certain Jacobi equations. The absence of conjugate points insures that a geodesic is a local minimizer and an isolated extremal of the variational problem [2].

**Definition 5** ([2]). Let  $\gamma$  be an abnormal geodesic on Engel manifold. A moment of time  $t$  is called conjugate if

$$e_*^{tX_2} \mathcal{D}_{\gamma(0)} = \mathcal{D}_{\gamma(t)}.$$

*Remark 4.* This definition is valid only for Engel manifolds. For the most general definition of conjugate points see [4]. For some particular cases see [2, 5].

**Theorem 5.** Let  $\gamma(t)$  be an abnormal curve on an Engel manifold and let

$$\Delta = T_6 + \frac{1}{2} \dot{T}_2 - \frac{1}{4} (T_2)^2.$$

If along this curve  $\Delta \leq 0$ , then arbitrary large pieces of  $\gamma(t)$  are  $C^0$ -local minimizing. If  $\Delta \geq C > 0$ , then  $\gamma(t)$  is not  $C^0$ -locally minimizing after  $t \geq \pi/\sqrt{C}$ .

*Proof.* Let us write down and analyse the corresponding Jacobi equation. Obviously  $e_*^{tX_2}(X_2(\gamma_0)) = X_2(\gamma_t)$ . So we must consider the evolution of  $a(t) = e_*^{tX_2} X_1$  along the abnormal curve  $\gamma_t$ . We have a conjugate point if and only if  $a(t)(\gamma(t)) \in \mathcal{D}_{\gamma(t)}$ . The corresponding differential equation on  $a(t)$  is

$$\dot{a}(t) = \frac{d}{dt} e_*^{tX_2} X_1 = e_*^{tX_2} [X_1, X_2] = [e_*^{tX_2} X_1, X_2] = [a(t), X_2].$$

Using canonical frame (14) we obtain

$$(25) \quad \begin{aligned} \dot{a}_1 &= -T_6 a_3 - X_1(T_6) a_4, \\ \dot{a}_2 &= -T_4 a_3 - X_1(T_4) a_4, \\ \dot{a}_3 &= a_1 - T_2 a_3 - (T_4 + X_1(T_2)) a_4, \\ \dot{a}_4 &= -T_2 a_4. \end{aligned}$$

Now restricting  $a(t)$  on the abnormal curve  $\gamma(t)$  we get a linear time-dependent system with boundary conditions  $a(0) = (1, 0, 0, 0)$  and  $a_3(t_1) = a_4(t_1) = 0$  where

$t_1$  is the supposed conjugate time. Note that the equations for  $a_1, a_3, a_4$  give a closed subsystem, so we don't need the second equation from (25). Moreover from the last equation and the boundary conditions we obtain  $a_4 \equiv 0$ . It remains to study the non trivial solutions to the boundary value problem

$$(26) \quad \begin{aligned} \dot{a}_1 &= -T_6 a_3, \\ \dot{a}_3 &= a_1 - T_2 a_3, \\ a_1(0) &= 1, a_3(0) = 0, a_3(t_1) = 0. \end{aligned}$$

Using that the abnormal curve is at least  $C^1$ -smooth, we rewrite (26) as a single second order ode:

$$\begin{aligned} \ddot{a}_3 + T_2 \dot{a}_3 + (T_6 + \dot{T}_2) a_3 &= 0 \\ a_3(0) &= 0, a_3(t_1) = 0, \dot{a}_3(0) = 1. \end{aligned}$$

This allows us to use powerful results from the oscillation theory of second order ODEs. After the change of variables

$$y = a_3 \exp \left( \int_0^t \frac{T_2(\tau)}{2} d\tau \right)$$

we get an equivalent formulation of the boundary value problem

$$\begin{aligned} \ddot{y} + \left( T_6 + \frac{\dot{T}_2}{2} - \frac{T_2^2}{4} \right) y &= 0 \\ y(0) &= 0, y(t) = 0, \dot{y}(0) = 1. \end{aligned}$$

Now the statement of the theorem is a direct consequence of the Sturm comparison theorem.  $\square$

Note that in the case of left-invariant structures, i.e. when all  $T_i$ 's are constants, we get a sharp result.

**Corollary 1.** *Let  $\gamma(t)$  be a strictly abnormal curve of a left-invariant Engel structure and let  $\Delta = T_6 - \frac{1}{4}(T_2)^2$ . If  $\Delta \leq 0$ , then arbitrary large pieces  $\gamma(t)$  are  $C^0$ -local minimizing. If  $\Delta > 0$ , then all the conjugate times are given by*

$$t_{conj} = \frac{\pi k}{\sqrt{\Delta}}, \quad \forall k \in \mathbb{Z}_+$$

and  $\gamma(t)$  is  $C^0$ -locally minimizing only on the interval  $(0, \pi/\sqrt{\Delta})$ .

In the next sections we have a closer look at the normal geodesics of the Hamiltonian system (22). We would like to find structures with the simplest possible geodesic flow. In particular we would like to find, when the Hamiltonian system is integrable. It will allow to write down an explicit parametrization of normal geodesics and to study the small spheres in the future.

#### 4. INTEGRABILITY OF LEFT-INVARIANT ENGEL STRUCTURES

Let us take a closer look at left-invariant Engel structures. A study of cojoint orbits gives an algebraic formalism to the integration of left-invariant Hamiltonian equations.

Let  $G$  be a finite-dimensional Lie group,  $\mathfrak{g}$  be its Lie algebra and  $\mathfrak{g}^*$  be its dual vector space. Consider the adjoint action  $\text{Ad}_g$  of  $G$  on  $\mathfrak{g}$ . The map dual to  $\text{Ad}_g$  defines the cojoint action on  $\mathfrak{g}^*$

$$\langle \text{Ad}_g^* \lambda, \xi \rangle = \langle \lambda, \text{Ad}_{g^{-1}} \xi \rangle, \quad \xi \in \mathfrak{g}, \lambda \in \mathfrak{g}^*.$$

Orbits of this action are called cojoint orbits of  $G$ . The coadjoint orbit  $\mathcal{O}_\lambda$  passing through a point  $\lambda \in \mathfrak{g}^*$  has an induced structure of a homogeneous space.

Under the action of  $G$  the stabilizer of  $\lambda$  is  $G_0 = \{g \in G \mid \text{Ad}_g^* \lambda = \lambda\}$ . The algebra of  $G_0$  is

$$\mathfrak{g}_0 = \{\xi \in \mathfrak{g} \mid \text{ad}_\xi^* \lambda = 0\}$$

where

$$\langle \text{ad}_\xi^* \lambda, \eta \rangle = -\langle \lambda, \text{ad}_\xi \eta \rangle.$$

The tangent space  $T_\lambda \mathcal{O}_\lambda$  is naturally associated with  $\mathfrak{g}/\mathfrak{g}_0$ . This allows to define the Kirillov symplectic form

$$\omega(\xi + \mathfrak{g}_0, \eta + \mathfrak{g}_0) = \langle \lambda, [\xi, \eta] \rangle.$$

The form  $\omega$  is well defined since

$$\langle \lambda, [\xi + \zeta, \eta] \rangle = -\langle \text{ad}_{\xi+\zeta}^* \lambda, \eta \rangle = -\langle \text{ad}_\xi^* \lambda, \eta \rangle = \langle \lambda, [\xi, \eta] \rangle, \quad \forall \zeta \in \mathfrak{g}_0.$$

**Definition 6.** A *Casimir function*  $F$  is an element of the center of the Poisson algebra, i.e.

$$\{F, H\} = 0, \quad \forall H \in C^\infty(\mathfrak{g}^*)$$

Casimir functions are  $\text{Ad}_g$  invariant [19], which implies that they are constant on the coadjoint orbits. Therefore the differential of  $F$  must annihilate  $T\mathcal{O}_\lambda$  and a non-trivial Casimir function can exist only if a generic coadjoint orbit has dimension less than  $\dim \mathfrak{g}$ . This is equivalent to the linear dependency of  $\text{ad}_{X_i}^*(\lambda)$  for generic  $\lambda \in \mathfrak{g}^*$  where  $X_i$  form the basis of the Lie algebra  $\mathfrak{g}$ . Equivalently if Pf defines a Pfaffian of a skew-symmetric matrix we must have

$$S = \text{Pf}(\text{ad}_{X_j}^*(\lambda), X_i) = \text{Pf}(\lambda, [X_i, X_j]) = \text{Pf} \omega(X_i, X_j) = \text{Pf}(\{h_i, h_j\}) = 0$$

Expressions for  $S$  are

$$\begin{aligned} I : S &= 0, \\ II : S &= T_2(T_3 h_3^2 + T_1 h_3 h_4 - h_4^2), \\ III : S &= 0, \\ IV : S &= -T_2 h_4, \\ V : S &= -\frac{T_2}{16(T_1)^2} (2T_1^2 h_3 + (T_1 h_2 - 2T_2 h_1)(T_1^2 + 4T_3) - 4T_1 h_4)^2 \end{aligned}$$

We see that structures I and III indeed can admit Casimir functions. Note that  $S = 0$  for II, IV and V only if  $T_2 = 0$ . In that case these families intersect with I or III. We are mainly interested in the case admitting strictly abnormal geodesics, i.e. case III. Therefore we give a detailed explanation for the case III and then sketch some results for the case I. Note that the family I and III have a non-empty intersection. Both classes have nice topological properties (see [16]).

The existence of at least one Casimir function in general will imply local integrability for a 4-dimensional Lie group. Indeed, coadjoint orbits are symplectic manifolds, therefore they are even-dimensional. This means that if there exists one Casimir function  $F$ , then locally there must exist another one  $G$  independent from  $F$ . These two Casimirs together with Hamiltonian of the problem  $H$  and a Hamiltonian of any right-invariant vector field will give us four independent commuting first integrals [19] unless  $H$  is a Casimir element itself.

**4.1. Integration for type I Lie algebras.** Lie algebras of type I have a simple structure. They are semi-direct products of a 3-dimensional abelian ideal  $V = \text{span}\{X_2, X_3, X_4\}$  and  $\mathfrak{gl}(1, \mathbb{R}) = \text{span}\{X_1\}$  acting on  $V$ . The structure of this Lie algebra is completely defined by the choice of scale for

$$A = \text{ad}|_V(X_1) = \begin{pmatrix} 0 & 0 & T_5 \\ 1 & 0 & T_3 \\ 0 & 1 & T_1 \end{pmatrix}$$

and its Jordan normal. Casimir functions for type I Lie algebras are invariants for the action of  $e^{tA}$  on  $V$ . We have different cases depending on the Jordan normal form of  $A$ . We summarize computations in Table 1. Here  $\tilde{h}_2, \tilde{h}_3, \tilde{h}_4$  denote Hamiltonian functions obtained from  $h_2, h_3, h_4$  by linear transformation which normalizes  $A$  to the Jordan form. Note that  $\text{rank } A$  is at least 2 therefore 2 of 3 functions in the 3rd column are always independent. Generic coadjoint orbits are level sets of those functions. We do not list in the table the degenerate cases, when some of  $h_i$  are equal to zero, but they can be easily deduced from the action of  $e^{tA}$ .

TABLE 1. Casimir functions for type I Lie algebra

$A$	Action of $e^{tA}$	Invariants
$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$	$\begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix}$	$\frac{\tilde{h}_3^{\lambda_1}}{\tilde{h}_2^{\lambda_2}}, \frac{\tilde{h}_4^{\lambda_2}}{\tilde{h}_3^{\lambda_3}}, \frac{\tilde{h}_2^{\lambda_3}}{\tilde{h}_4^{\lambda_1}}$
$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} e^{t\lambda_1} & te^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_1} & 0 \\ 0 & 0 & e^{t\lambda_2} \end{pmatrix}$	$\frac{\tilde{h}_3^{\lambda_2}}{\tilde{h}_4^{\lambda_1}}, \lambda_1 \frac{\tilde{h}_2}{\tilde{h}_3} - \ln \tilde{h}_3, \lambda_2 \frac{\tilde{h}_2}{\tilde{h}_3} - \ln \tilde{h}_4$
$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} e^{t\lambda} & te^{t\lambda} & \frac{t^2}{2}e^{t\lambda} \\ 0 & e^{t\lambda} & te^{t\lambda} \\ 0 & 0 & e^{t\lambda} \end{pmatrix}$	$\lambda \frac{\tilde{h}_3}{\tilde{h}_4} - \ln \tilde{h}_4, \frac{2\tilde{h}_2}{\tilde{h}_4} - \left(\frac{\tilde{h}_3}{\tilde{h}_4}\right)^2$
$\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} e^{at} \cos bt & -e^{at} \sin bt & 0 \\ e^{at} \sin bt & e^{at} \cos bt & 0 \\ 0 & 0 & e^{t\lambda} \end{pmatrix}$	$\frac{(\tilde{h}_2^2 + \tilde{h}_3^2)^{\frac{\lambda}{2}}}{\tilde{h}_4^a}, \lambda \arctan\left(\frac{\tilde{h}_3}{\tilde{h}_2}\right) - b \ln(\tilde{h}_4),$ $a \arctan\left(\frac{\tilde{h}_3}{\tilde{h}_2}\right) - \frac{b}{2} \ln(\tilde{h}_2^2 + \tilde{h}_3^2)$

One particularly nice example of this type is given in the next proposition.

**Proposition 5.** *For any  $m > n \geq 0$  there exists an integrable Engel model from family I which admits two polynomial left-invariant first integrals of orders  $n + 1$  and  $m + 1$ .*

*Proof.* Consider Lie algebra of type I with the following invariants

$$\begin{aligned} T_1 &= n + m - 1, \\ T_3 &= n + m - nm, \\ T_6 &= -nm. \end{aligned}$$

Then the eigenvalues are  $\lambda_1 = n, \lambda_2 = m, \lambda_3 = -1$  and two Casimir functions have the form

$$\begin{aligned} F_1 &= \tilde{h}_2 \tilde{h}_3^m = \left( \frac{h_3 + h_4 - (h_2 + h_3)n}{(1+m)(m-n)} \right) \left( \frac{h_4 + mn h_2 - (m+n)h_3}{(1+m)(1+n)} \right)^m \\ F_2 &= \tilde{h}_1 \tilde{h}_3^n = \left( \frac{m(h_2 + h_3) - h_3 - h_4}{(1+n)(m-n)} \right) \left( \frac{h_4 + mn h_2 - (m+n)h_3}{(1+m)(1+n)} \right)^n \end{aligned}$$

Since the change of coordinates  $h_i \mapsto \tilde{h}_i$  is a non-degenerate linear map the integrals are functionally independent. Thus we obtain an integrable structure with polynomial first integrals of orders  $n + 1$  and  $m + 1$ .  $\square$

**4.2. Integration for type III Lie algebras.** Any algebra of type III is a central extension of 3-dimensional Lie algebra. The center element is

$$X'_4 = X_4 + T_4 X_1 - T_3 X_2.$$

The underlying lie algebra is semi-simple iff  $D = (T_4)^2 + T_3 T_6 \neq 0$ . If  $D < 0$  and  $T_3 < 0$  (equivalently  $T_6 < 0$ ) then it is  $\mathfrak{so}(3, \mathbb{R})$ . Otherwise it is split real form  $\mathfrak{sl}(2, \mathbb{R})$ .

If  $T_4 = T_6 = 0$  then we have a trivial extension either of the Lie algebra of Euclidean motions of the plane ( $T_3 > 0$ ) or the Lie algebra of Poincaré motions of the plane ( $T_3 < 0$ ).

Finally in the case  $D = 0$ ,  $T_4 \neq 0$  we obtain a non-trivial extension of arbitrary solvable Lie algebra of dimension 3 with 2-dimensional derived algebra.

Sub-Riemannian Engel structures of type III are in fact super-integrable.

**Proposition 6.** *The Hamiltonian system of type III is superintegrable, meaning that it has four independent commuting first integrals including the Hamiltonian  $H$  and one more independent first integral that commutes with  $H$ .*

*Proof.* Let us use  $X'_4$  instead of  $X_4$  for all computations in this subsection. Then

$$(27) \quad [X_1, X_2] = X_3,$$

$$(28) \quad [X_1, X_3] = X'_4 - T_4 X_1 + T_3 X_2,$$

$$(29) \quad [X_2, X_3] = T_6 X_1 + T_4 X_2$$

and therefore  $h_4 = \langle \lambda, X'_4 \rangle$  is a Casimir function.

In the basis  $X_1, X_2, X_3, X'_4$  the Hamiltonian system takes the form

$$(30) \quad \begin{aligned} \dot{h}_1 &= -h_2 h_3, \\ \dot{h}_2 &= h_1 h_3, \\ \dot{h}_3 &= h_1 h_4 - T_4(h_1^2 - h_2^2) + (T_3 + T_6)h_1 h_2, \end{aligned}$$

where  $h_4 = \text{const}$ . It is easy to see that it has the following first integral

$$G = \frac{h_3^2}{2} - h_4 h_2 + \frac{T_3 + T_6}{4}(h_1^2 - h_2^2) + T_4 h_1 h_2,$$

which comes from the Casimir function

$$\tilde{G} = G + \frac{T_6 - T_3}{2}H.$$

Two more independent first integrals is a couple of two right-invariant Hamiltonians. Indeed, let  $I : g \mapsto g^{-1}$  be the involution mapping and  $X^R(g) = I_* X^L(g)$  be the right invariant fields constructed from the left-invariant ones. Let  $h_i^R$  be the right-invariant Hamiltonians. We know that  $h_i^R$  commute with any left-invariant Hamiltonian and that  $G, h_4$  are Casimir function. Therefore  $H, G, h_4, h_1^R$  are commuting and  $\{H, h_2^R\} = 0$ .

We claim that  $dH, dG, dh_4, dh_1^R, dh_2^R$  are linearly independent almost everywhere. Since we have a transitive action of our Lie group, it is enough to check the linear independence at the identity. Assume that left-invariant Hamiltonians and right invariant Hamiltonians are related by

$$X_i^R(g) = \alpha_i^j(g) X_j^L(g) \quad \Rightarrow \quad h_i^R = \alpha_i^j(g) h_j,$$

where  $\alpha_i^j(\text{id}) = -\delta_i^j$ . In [20] it was shown, that in the coordinates of the first kind

$$\frac{\partial \alpha_i^k}{\partial x^j}(\text{id}) = c_{ji}^k.$$

Using this we obtain

$$\begin{pmatrix} dH \\ dG \\ dh_4 \\ dh_1^R \\ dh_2^R \end{pmatrix} = \begin{pmatrix} h_1 & h_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{T_3+T_6}{2}h_1+T_4h_2 & -h_4-\frac{T_3+T_6}{2}h_2+T_4h_1 & h_3-h_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -h_3 & -h_4+T_4h_1-T_3h_2 & 0 \\ 0 & -1 & 0 & 0 & h_3 & 0 & -T_6h_1-T_4h_2 & 0 \end{pmatrix}$$

The determinant of the first, third, fourth, fifth and sixth rows is equal to  $h_1 h_3^3$ . Therefore  $H, G, h_4, h_1^R, h_2^R$  are almost everywhere functionally independent first integrals.  $\square$

Now we can integrate the vertical part of the Hamiltonian system. Since different values of  $H > 0$  correspond to the same reparameterized geodesics, we can just consider the level set  $H = 1/2$ . Then solutions of our system lie on the intersection of two quadrics

$$\begin{aligned} Q_1 : h_1^2 + h_2^2 - 1 &= 0, \\ Q_2 : 2h_3^2 - 4h_4h_2 + 4T_4h_1h_2 + (T_3 + T_6)(h_1^2 - h_2^2) - 4G &= 0. \end{aligned}$$

It is well known that an intersection of two quadrics is an elliptic curve  $\Gamma$  [14]. We are going to find a parameterization for  $\Gamma$ . Consider a rational parameterization of the cylinder  $Q_1$ :

$$h_1 = \frac{1 - \lambda^2}{1 + \lambda^2}, \quad h_2 = \frac{2\lambda}{1 + \lambda^2}, \quad h_3 = \frac{y}{\sqrt{2}(1 + \lambda^2)}.$$

Applying this transform to  $Q_2$  we obtain a parameterization of  $\Gamma$ :

$$(31) \quad y^2 = (4G - T_3 - T_6) - 8(T_4 - h_4)\lambda + (8G + 6(T_3 + T_6))\lambda^2 + 8(h_4 + T_4)\lambda^3 + (4G - T_3 - T_6)\lambda^4 = P(\lambda)$$

Substituting this into any equation (30) gives us

$$(32) \quad \dot{\lambda}^2 = P(\lambda)$$

i.e. the solution of this equation is given by the inversion of an elliptic integral of the first kind.

If  $4G - T_3 - T_6 = 0$  and  $P(\lambda)$  has no double roots, then by a change of variables

$$w = y\sqrt{2(h_4 + T_4)}, \quad z = \lambda + \frac{4G + 3(T_3 + T_6)}{12(h_4 + T_4)}.$$

we put the curve (31) into it's Weierstrass normal form

$$w^2 = 4z^3 + g_2z + g_3.$$

If we introduce a new time parameter

$$\tau = t\sqrt{2(h_4 + T_4)}$$

then (32) becomes

$$(z')^2 = 4z^3 + g_2z + g_3.$$

Solution of this equation is now given by the Weierstrass elliptic function

$$z(t) = \wp\left(t\sqrt{2(h_4 + T_4)}, g_2, g_3\right).$$

Note that the expression under the square root can be negative, but  $\wp(it, g_2, g_3) = -\wp(t, g_2, -g_3)$  and we can reparameterize it using real time.

Assume now that  $P(\lambda)$  has no double roots and  $4G - T_3 - T_6 \neq 0$ , then we proceed as in [7]. First we take a change of variables

$$w = y\sqrt{4G - T_3 - T_6}, \quad z = \lambda + \frac{2(h_4 + T_4)}{4G - T_3 - T_6}.$$

Then (31) becomes

$$w^2 = z^4 - 6Az^2 + 4Bz + C$$

and by a change of time variable

$$\tau = t\sqrt{4G - T_3 - T_6}$$

equation (32) is replaced by

$$(33) \quad (z')^2 = z^4 - 6Az^2 + 4Bz + C.$$



To solve this equation, we need a clever parameterization for the elliptic curve  $\Gamma$ . Such a parameterization is given by [7]

$$\begin{aligned} z &= \zeta\left(u + \frac{v}{2}\right) - \zeta\left(u - \frac{v}{2}\right) - \zeta(v), \\ w &= \wp\left(u - \frac{v}{2}\right) - \wp\left(u + \frac{v}{2}\right), \end{aligned}$$

where  $\zeta(u, g_2, g_3)$  is the Weierstrass zeta function,

$$g_2 = C + 3A^2, \quad g_3 = -AC + A^3 - B^2$$

are the invariants of our curve  $\Gamma$  and  $v$  can be determined from

$$A = \wp(v, g_2, g_3), \quad B = \wp'(v, g_2, g_3).$$

Since  $\zeta'(u) = -\wp(u)$ , we see that

$$\frac{dz}{du} = w.$$

From this it obviously follows

$$w^2 = (z')^2 = \left(\frac{dz}{du}\right)^2 (u')^2 = (u')^2 w^2 \Rightarrow (u')^2 = 1.$$

So the solution of (33) can be written as

$$\begin{aligned} z(t) &= \zeta\left(\sqrt{4G - T_3 - T_6}t + \frac{v}{2}, g_2, g_3\right) - \\ &\quad - \zeta\left(\sqrt{4G - T_3 - T_6}t - \frac{v}{2}, g_2, g_3\right) - \\ &\quad - \zeta(v, g_2, g_3). \end{aligned}$$

All the singular solutions, i.e. when  $P(\lambda)$  has some double roots, are given by the limits of the corresponding non-singular solutions.

*Remark 5.* There is a couple of other standard ways to obtain a parameterization for solutions of our system. For example, one could use the same techniques, that Volterra used for integration of the gyrostat equations in terms of the Weierstrass sigma functions [24]. But this approach is rather indirect, since it requires knowledge of roots of some fourth order polynomial. An alternative procedure for the integration of gyrostat equations was given in [26]. Secondly, we could've used a different parametrization of our elliptic curve (31). For example, a birational transformation due to Biermann [25] would give us a solution of (32) in a form

$$\lambda = \lambda_0 + \frac{\sqrt{P(\lambda_0)}\wp(t) + \frac{1}{2}P'(\lambda_0)\left(\wp(t) - \frac{1}{24}P''(\lambda_0)\right) + \frac{1}{24}P(\lambda_0)P'''(\lambda_0)}{2\left(\wp(t) - \frac{1}{24}P''(\lambda_0)\right)^2 - \frac{1}{48}P(\lambda_0)P^{(4)}(\lambda_0)}$$

which is called the Biermann-Weierstrass formula.

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